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Growth and Complete Sequences of Generalized Axisymmetric Potentials*

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1. INTRODUCTION

Solutions of the n-dimensional Laplace equation

$$\frac{\partial^2 H}{\partial {x_1}^2} + \frac{\partial^2 H}{\partial {x_2}^2} + \cdots + \frac{\partial^2 H}{\partial {x_n}^2} = 0$$

which depend only on the variables

$$x = x_1 \qquad \text{and} \qquad y = ({x_2}^2 + {x_3}^2 + \cdots + {x_n}^2)^{1/2}$$

are naturally called axisymmetric potentials. These satisfy the partial differential equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{2\mu}{y}\frac{\partial H}{\partial y} = 0 \tag{1.0}$$

where $2\mu = n - 2$.

Solutions of Eq. (1.0) for $2\mu > 0$ (2μ not necessarily an integer) were first investigated by Weinstein [7] and called *generalized axisymmetric potentials.*

Let $H(x, y)$ be a generalized axisymmetric potential (GASP). If H is entire, that is, it has no finite singularities, then Gilbert's A_μ integral operator [2, p. 168] transforms an entire function h of a single complex variable to H:

$$\begin{aligned} H(x, y) &= A_\mu(h) \\ &= \alpha_\mu \int_L h(z)(\zeta - \zeta^{-1})^{2\mu}\, \zeta^{-1}\, d\zeta \end{aligned} \tag{1.1}$$

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where

$$z = x + iy(\zeta + \zeta^{-1})/2, \qquad L = \{e^{i\phi}: 0 \leqslant \phi \leqslant \pi\}$$

and

$$\alpha_\mu = \left[\int_L (\zeta - \zeta^{-1})^{2\mu-1}\, \zeta^{-1}\, d\zeta\right]^{-1}$$
$$= 4\Gamma(2\mu)/(4i)^{2\mu}\, \Gamma(\mu)^2.$$

For H entire, the integral representation (1.1) holds throughout the plane.

The function h is called the A_μ associate of H. Letting $x = r\cos\theta$, $y = r\sin\theta$, an entire GASP has expansion

$$H(r, \theta) = \sum_{n=0}^{\infty} a_n r^n C_n^\mu(\cos\theta) \tag{1.2}$$

which converges uniformly on compact sets, where C_n^μ are the Gegenbauer polynomials of degree n. The corresponding A_μ associate is then

$$h(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\mu)}{\Gamma(2\mu)\, \Gamma(n+1)}\, a_n z^n.$$

The inverse transformation A_μ^{-1} is given by

$$h(z) = A_\mu^{-1}(H)$$
$$= \int_0^\pi H(r, \theta)\, K(z/r, \theta)\, d\theta, \qquad |z| < r, \tag{1.3}$$

where

$$K(z/r, \theta) = \frac{\mu\Gamma(2\mu)}{2^{2\mu-1}\Gamma(\mu + 1/2)^2}\, \frac{(\sin\theta)^{2\mu}(1 - z^2/r^2)}{[1 - 2(z/r)\cos\theta + z^2/r^2]^{\mu+1}}$$

[2, p. 173].

The A_μ integral operator provides a means of obtaining function theoretic type results for GASP's. That is, although function theoretic techniques may not have analogs in the theory of partial differential equations, results may be obtained directly by using the A_μ integral operator to relate properties of a GASP and its associate. In this paper the relation of the growth of an entire GASP to that of its A_μ associate will be considered, with the primary aim of characterizing the growth of the GASP in terms of its coefficients a_n in the expansion (1.2).

Section 2 deals with order, and functions of regular growth. In Section 3 proximate orders for GASP's, type with respect to a proximate order, and functions of perfectly regular growth are considered. In Section 4 we apply the previous results to obtain a method for generating complete sequences of GASP's from a single entire GASP.

2. Order of an Entire GASP

Let $M(r, H) = \max_\theta |H(r, \theta)|$. We define the order ρ_H of the GASP H just as is done for entire analytic functions of a single complex variable:

$$\rho_H = \limsup_{r\to\infty} \frac{\log\log M(r, H)}{\log r}.$$

From the A_μ integral operator (1.1) we have

$$M(r, H) \leqslant M(r, h). \tag{2.1}$$

Letting $z = \lambda re^{it}$, $0 < \lambda < 1$, in the inverse transform A_μ^{-1} yields

$$\begin{aligned} M(\lambda r, h) &= \max_t \left| \int_0^\pi H(r, \theta)\, K(\lambda e^{it}, \theta)\, d\theta \right| \\ &\leqslant \max_t \max_\theta \pi\, |H(r, \theta)\, K(\lambda e^{it}, \theta)| \\ &\leqslant \pi \max_{\theta,t} K(\lambda e^{it}, \theta)\, M(r, H). \end{aligned}$$

Thus

$$M(r, h) \leqslant k(\lambda)\, M(\lambda^{-1}r, H) \tag{2.2}$$

where

$$k(\lambda) = \frac{\pi\mu\Gamma(2\mu)}{2^{2\mu-1}\Gamma(\mu+1/2)^2} \max_{\theta,t} \left| \frac{(\sin\theta)^{2\mu}(1-\lambda^2 e^{2it})}{(1-2\lambda e^{it}\cos\theta + \lambda^2 e^{2it})^{\mu+1}} \right|.$$

We note that $\lim_{\lambda\to 1} k(\lambda) = \infty$.

Theorem 2.1. *Let H be an entire* GASP *and let h be its A_μ associate. Then the orders ρ_H of H and ρ_h of h are equal.*

Proof. Equation (2.1) implies $\rho_H \leqslant \rho_h$. Further, by (2.2) we have

$$\begin{aligned} \rho_h &= \limsup_{r\to\infty} \frac{\log\log M(r, h)}{\log r} \\ &\leqslant \limsup_{r\to\infty} \frac{\log\log k(\lambda)\, M(\lambda^{-1}r, H)}{\log r} \\ &= \limsup_{r\to\infty} \frac{\log[\log M(\lambda^{-1}r, H) + \log k(\lambda)]}{\log \lambda^{-1}r - \log\lambda^{-1}} \\ &= \limsup_{r\to\infty} \frac{\log\log M(\lambda^{-1}r, H)}{\log\lambda^{-1}r} \\ &= \rho_H. \end{aligned}$$

COROLLARY 2.2. *Let H be an entire* GASP *and* $\{a_n\}_0^\infty$ *the coefficients of H in its expansion* (1.2). *Then*

$$\rho_H = \limsup_{n\to\infty} \frac{n \log n}{\log | a_n |^{-1}}.$$

Proof. The order of an entire function of a single complex variable $\sum_{n=0}^{\infty} c_n z^n$ is given in terms of its coefficients by [5, p. 4]

$$\limsup_{n\to\infty} \; (n \; \log n)/\log | c_n |^{-1}.$$

The GASP (1.2) has A_μ associate

$$h(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2\mu)}{\Gamma(2\mu)\,\Gamma(n+1)}\, a_n z^n,$$

thus

$$\rho_h = \limsup_{n\to\infty} (n \log n) \left[\log | a_n |^{-1} + \log \frac{\Gamma(2\mu)\,\Gamma(n+1)}{\Gamma(n+2\mu)}\right]^{-1}.$$

Since $[\Gamma(2\mu)\,\Gamma(n+1)/\Gamma(n+2\mu)]^{1/n} \to 1$,

$$\rho_h = \limsup_{n\to\infty} \frac{n \log n}{\log | a_n |^{-1}}$$

which by Theorem 2.1 equals ρ_H.

An entire GASP H will be said to have *regular growth* if

$$\lim_{r\to\infty} \frac{\log \log M(r, H)}{\log r}$$

exists (cf. [6, p. 41] regarding regular growth of entire functions of a single complex variable). Thus a GASP H of order $\rho_H = \rho < \infty$ is of regular growth if and only if for every $\epsilon > 0$, there exists R such that

$$\exp(r^{\rho-\epsilon}) \leqslant M(r, H) \leqslant \exp(r^{\rho+\epsilon})$$

for every $r \geqslant R$.

THEOREM 2.3. *Let H be an entire* GASP *and* $\{a_n\}_0^\infty$ *the coefficients of H in the expansion* (1.2). *Then*

$$\lim_{r\to\infty} \frac{\log \log M(r, H)}{\log r} = \rho$$

if and only if for every $\epsilon > 0$,

$$\frac{n \log n}{\log | a_n |^{-1}} \leqslant \rho + \epsilon$$

for all n sufficiently large, and there exists a sequence $\{n_k\}_{k=0}^{\infty}$ *such that*

$$\lim_{k\to\infty} \frac{\log n_{k+1}}{\log n_k} = 1$$

for which

$$\lim_{k\to\infty} \frac{n_k \log n_k}{\log | a_n |^{-1}} = \rho.$$

Proof. Using the inequalities (2.1) and (2.2) a routine computation verifies that an entire GASP is of regular growth if and only if its associate is. Further an entire function of a single complex variable $\sum_{n=0}^{\infty} a_n z^n$ is of regular growth if and only if its coefficients satisfy the conditions stated in the above theorem (cf. [6, p. 44]). Therefore since the A_μ associate of

$$H(r, \theta) = \sum_{n=0}^{\infty} a_n r^n C_n^{\mu}(\cos \theta)$$

is

$$h(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\mu)}{\Gamma(2\mu)\, \Gamma(n + 1)} a_n z^n,$$

the result follows as a consequence of the fact that

$$\lim_{n\to\infty} \left[\frac{\Gamma(n + 2\mu)}{\Gamma(2\mu)\, \Gamma(n)}\right]^{1/n} = 1.$$

3. Proximate Order and Respective Type of an Entire GASP

The concept of proximate order generalizes that of order, and is introduced in function theory so as to obtain a measure of growth more refined than type (eg. see [5, p. 32]). We define proximate orders for entire GASP's following the usual function theoretic definition.

Definition 3.1. Let H be an entire GASP of order $\rho \neq 0, \infty$. A continuous function $\rho(r)$ which is differentiable except perhaps at isolated points is said to be a *proximate order* for H if it satisfies

$$\lim_{r\to\infty} \rho(r) = \rho, \tag{3.1}$$

$$\lim_{r\to\infty} r\rho'(r) \log r = 0, \tag{3.2}$$

$$\limsup_{r\to\infty} \frac{\log M(r, H)}{r^{\rho(r)}} \neq 0, \infty. \tag{3.3}$$

The quantity

$$\sigma_H = \limsup_{r\to\infty} \frac{\log M(r, H)}{r^{\rho(r)}}$$

is called the *type of H with respect to the proximate order* $\rho(r)$.

For entire functions of a single complex variable, proximate orders always exist but are not unique [1, 5]. The questions of the existence and uniqueness of proximate orders for entire GASP's and the relation of proximate orders and respective types to those of their corresponding A_μ associates are answered by

THEOREM 3.2. *Let H be an entire* GASP *of order* $\rho \neq 0, \infty$ *and h its* A_μ *associate. Then a function* $\rho(r)$ *is a proximate order for H if and only if it is a proximate order for the associate h. Further the types* σ_H *of H and* σ_h *of h with respect to the same proximate order* $\rho(r)$ *are equal.*

Proof. Let $\rho(r)$ be a continuous function which is differentiable except perhaps at isolated points and satisfies (3.1) and (3.2). The inequality (2.2) implies

$$\limsup_{r\to\infty} \frac{\log M(r, h)}{r^{\rho(r)}} \leqslant \limsup_{r\to\infty} \frac{\log k(\lambda)\, M(\lambda^{-1}r, H)}{r^{\rho(r)}}$$

$$= \limsup_{r\to\infty} \frac{\log M(\lambda^{-1}r, H)}{r^{\rho(r)}}.$$

Now the properties (3.1) and (3.2) imply the function $L(r) = r^{\rho(r)-\rho}$ is slowly increasing. That is, for $\alpha > 0$ $\lim_{r\to\infty} L(\alpha r)/L(r) = 1$. Thus given $\epsilon > 0$ there exists R such that

$$(\lambda^{-1}r)^{\rho(\lambda^{-1}r)} < (1+\epsilon)\, \lambda^{-\rho} r^{\rho(r)}, \qquad r \geqslant R.$$

Therefore

$$\limsup_{r\to\infty} \frac{\log M(\lambda^{-1}r, H)}{r^{\rho(r)}} \leqslant \limsup_{r\to\infty} \frac{(1+\epsilon)\, \lambda^{-\rho} \log M(\lambda^{-1}r, H)}{(\lambda^{-1}r)^{\rho(\lambda^{-1}r)}}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log M(r, h)}{r^{\rho(r)}} \leqslant \limsup_{r\to\infty} \frac{(1+\epsilon)\, \lambda^{-\rho} \log M(r, H)}{r^{\rho(r)}}.$$

Since the choice of $\epsilon > 0$ and $\lambda \in (0, 1)$ are arbitrary, this yields

$$\limsup_{r\to\infty} \frac{\log M(r, h)}{r^{\rho(r)}} \leqslant \limsup_{r\to\infty} \frac{\log M(r, H)}{r^{\rho(r)}}.$$

By (2.1) the reverse inequality is immediate. Further, by Theorem 2.1 the order of H and of its A_μ associate h are equal. Thus the result follows from the equality

$$\limsup_{r\to\infty} \frac{\log M(r, h)}{r^{\rho(r)}} = \limsup_{r\to\infty} \frac{\log M(r, H)}{r^{\rho(r)}}.$$

As a corollary we have a formula expressing the type of a GASP with respect to a proximate order in terms of the coefficients in its expansion (1.2):

COROLLARY 3.3. *Let H be an entire* GASP *of order ρ and let $\{a_n\}_0^\infty$ be the coefficients of H in its expansion* (1.2). *If $\rho(r)$ is a proximate order for H and σ is the type of H with respect to $\rho(r)$, then*

$$(\sigma\rho e)^{1/\rho} = \limsup_{r\to\infty} \phi(n) \mid a_n \mid^{1/n}, \tag{3.4}$$

where $\phi(t)$ is the inverse of the function $t = r^{\rho(r)}$.

Proof. Equation (3.4) gives the type of an entire function of a single complex variable $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with respect to the proximate order $\rho(r)$, [5, p. 42]. The proof now follows that of Corollary 2.2.

Gilbert [3] has shown that the type

$$\tau = \limsup \frac{\log M(r, H)}{r^\rho} \qquad (\rho = \text{order } H),$$

of an entire GASP equals the type of its A_μ associate, and as a consequence

$$(\tau e\rho)^{1/\rho} = \limsup_{n\to\infty} n^{1/\rho} \mid a_n \mid^{1/n}.$$

Since proximate order and type with respect to a proximate order generalize order and type, these results obtain as special cases of Theorem 3.2 and Corollary 3.3, i.e., the case in which $\rho(r) \equiv \rho$.

We call GASP's of order ρ for which the limit

$$\lim_{r\to\infty} \frac{\log M(r, H)}{r^\rho}$$

exists, functions of *perfectly regular growth.* From the necessary and sufficient conditions that an entire analytic function be of perfectly regular growth [6, p. 44], we have on arguing as in Theorem 2.3 the following characterization of entire GASP's of perfectly regular growth.

THEOREM 3.4. *Let H be an entire* GASP *of order ρ and let $\{a_n\}_0^\infty$ be its coefficients in the expansion* (1.2), *Then,*

$$\lim_{r\to\infty} \frac{\log M(r, H)}{r^\rho} = \tau$$

if only if for every $\epsilon > 0$

$$n \mid a_n \mid^{\rho/n} < \rho e(\tau + \epsilon)$$

for all n sufficiently large and there exists a sequence $\{n_k\}_0^\infty$ *such that*

$$\lim_{k\to\infty} n_{k+1}/n_k = 1$$

for which

$$\lim_{k\to\infty} n_k \mid a_{n_k} \mid^{\rho/n_k} = \rho e.$$

4. Generating Complete Sequences of GASP's

As an application of the preceding results, we obtain a method for generating complete sequences of GASP's from a single entire GASP. This will be done by appealing to an analogous function theoretic result first proved by Gel'fond [4], and later refined by Levin [5, p. 217].

Inspection of the partial differential equation (1.0) shows that entire GASP's are invariant under homothetic transformation. That is, if $H(x, y)$ is an entire GASP and λ is any real number, then $H(\lambda x, \lambda y)$ is an entire GASP. Defining $C_n^\mu(x, y) = r^n C_n^\mu(\cos\theta)$ we have.

$$\begin{aligned} C_n^\mu(x, y) &= A_\mu(z^n) \\ &= \alpha_\mu \int_0^\pi (x + iy\cos t)^n (\sin t)^{2\mu-1}\, dt. \end{aligned}$$

Thus if H has the expansion (1.2), $H(\lambda x, \lambda y) = \sum_{n=0}^\infty a_n \lambda^n C_n^\mu(x, y)$.

Theorem 4.1. *Let* $H(x, y) = \sum_{k=0}^\infty a_k C_k^\mu(x, y)$ *be an entire real-valued* GASP *with* $a_k \neq 0$, $k = 0, 1, 2, \ldots$ *If* H *has type at most* σ *with respect to the proximate order* $\rho(r)$, *and* $\{\lambda_n\}_0^\infty$ *is any sequence of distinct real numbers, then the sequence of entire* GASP's

$$H_n(x, y) = H(\lambda_n x, \lambda_n y), \qquad n = 0, 1, 2\ldots$$

is complete in the space of all real-valued GASP's *on the open disk* D_R *centered at* 0 *and of radius* R *given by*

$$R^\rho = \frac{1}{e\rho\sigma} \limsup_{n\to\infty} \frac{n}{\mid \lambda_n \mid^{\rho(\mid\lambda_n\mid)}}. \tag{4.1}$$

Proof. By Theorem 3.2 $\rho(r)$ is a proximate order for the A_μ associate of H,

$$h(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\mu)}{\Gamma(2\mu)\, \Gamma(k + 1)}\, a_k z^k,$$

and the types with respect to $\rho(r)$ are equal i.e., $\sigma_h = \sigma_H = \sigma$. Thus by the function theoretic result [5, p. 217], the sequence of functions

$$h_n(z) = h(\lambda_n z)$$

is complete on the open disk D_R where R is given by (4.1).

Now, let $G(x, y)$ be any real-valued GASP regular on D_R, and g its A_μ associate. Then g is analytic on D_R and by the completeness of $\{h_k(z)\}_0^\infty$ there exist linear combinations

$$q_n(z) = \sum_{k=0}^{n} a_k^{(n)} h_k(z)$$

which converge uniformly to g on compact subsets of D_R. Now,

$$\begin{aligned} A_\mu(h_n) &= \alpha_\mu \int_0^\pi h_n(x + iy \cos t)(\sin t)^{2\mu-1}\, dt \\ &= \alpha_\mu \int_0^\pi h(\lambda_n x + i\lambda_n y \cos t)(\sin t)^{2\mu-1}\, dt \\ &= H(\lambda_n x, \lambda_n y) \\ &= H_n(x, y). \end{aligned}$$

Thus letting

$$Q_n(x, y) = \sum_{k=0}^{n} a_k^{(n)} H_k(x, y)$$

yields

$$\begin{aligned} &| G(x, y) - Q_n(x, y)| \\ &\quad = | A_\mu(g - q_n)| \\ &\quad = \left| \alpha_\mu \int_0^\pi [g(x + iy \cos t) - q_n(x + iy \cos t)](\sin t)^{2\mu-1}\, dt \right| \\ &\quad \leqslant \max_t | g(x + iy \cos t) - q_n(x + iy \cos t)|. \end{aligned}$$

Thus the completeness of $\{h_n\}_0^\infty$ on D_R implies that of $\{H_n\}_0^\infty$.

The result of Theorem 4.1 also holds for complex-valued GASP's as is evident by arguing on real and imaginary parts.

Thus a single entire solution of Eq. (1.0) can be used to construct a sequence of solutions which is complete in the space of all solutions regular on a disk D_R of arbitrary radius R. This is done merely by appropriate choice of the numbers λ_n. For example, choosing $\{\lambda_n\}_0^\infty$ as any bounded sequence of distinct reals yields $R = \infty$ in Eq. (4.1) so that $\{H_n\}_0^\infty$ forms a complete sequence over the entire plane.

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